(1) Vibrating String Problem. given. $U(0,t) = U(L,t) = 0$ B.C

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 U}{\partial t^2}$$

$$\left.\begin{aligned} U(x,0) &= f(x) \\ \frac{\partial U(x,0)}{\partial t} &= 0 \end{aligned}\right\} \text{I.C.}$$

$U(x,t) = ?$

Solution: $U(x,t) = X(x)\,T(t)$

subst. in eq $\Rightarrow$ $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = \dfrac{1}{a^2T}\dfrac{d^2T}{dt^2} = \alpha = -\lambda^2$

Solving eqs.

$$\left.\begin{aligned} X(x) &= C_1 \cos\lambda x + C_2 \sin\lambda x \\ T(t) &= C_3 \cos a\lambda t + C_4 \sin a\lambda t \end{aligned}\right.$$

① $U(0,t) = 0$; $X(0)\,T(t) = 0 \Rightarrow X(0) = 0 \Rightarrow C_1 = 0$

② $U(L,t) = 0$; $X(L)\,T(t) = 0$ $\quad \sin\lambda L = 0 = \sin n\pi \quad n = 1, 2, \ldots$

$X(L) = 0 = C_2 \sin\lambda L \Rightarrow \sin\lambda L = \sin n\pi \Rightarrow \lambda = \dfrac{n\pi}{L}$

③ $\dfrac{\partial}{\partial t}U(x,0) = 0 \Rightarrow \dfrac{\partial T(0)}{\partial t}X(x) = 0 \Rightarrow \left.(-a\lambda C_3 \sin a\lambda t + a\lambda C_4 \cos a\lambda t)\right|_{t=0} X(x) = 0$

$C_4 = 0$

$X(x) = C_2 \sin\lambda x$, $\quad T(t) = C_3 \cos a\lambda t$

$U(x,t) = X(x)\,T(t) = C_2 \sin\dfrac{n\pi x}{L}\, C_3 \cos\dfrac{n\pi a t}{L} \qquad n = 1, 2, \ldots$

$U(x,t) = \displaystyle\sum_{n=1}^{\infty} \underbrace{C_2C_3}_{C_n} \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$

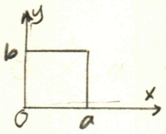
$U(x,t) = \displaystyle\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$

④ $U(x,0) = f(x) = \displaystyle\sum_{n=0}^{\infty} C_n \sin\frac{n\pi x}{L} \Rightarrow C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$

$$U(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\right)\sin\frac{n\pi x}{L}\cos\frac{n\pi a t}{L}$$

Ex19: Steady State temp. dist. in a slab.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



$0<x<a$

$0<y<b$

given, $T(0,y) = T(a,y) = 0$

$T(x,0) = f(x)$

$T(x,b) = 0$

$$T(x,y) = X(x)\,Y(y) \Rightarrow Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda^2$$

$$X(x) = C_1 \sin\lambda x + C_2 \cos\lambda x$$

$$Y(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$$

① $T(0,y) = 0 \Rightarrow X(0) = 0 \Rightarrow C_2 = 0$

② $T(a,y) = 0 \Rightarrow X(a) = 0 \Rightarrow C_1 \sin\lambda a = 0 = \sin n\pi \Rightarrow \lambda = \frac{n\pi}{a}$

$X(x) = C_1 \sin\frac{n\pi x}{a}$, $Y(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$

③ $T(x,0) = f(x) \Rightarrow T(x,y) = C_1 \sin\frac{n\pi x}{a}\left[C_3 e^{\lambda y} + C_4 e^{-\lambda y}\right]$

④ $T(x,b) = 0 \Rightarrow C_3 e^{\lambda b} + C_4 e^{-\lambda b} = 0 \Rightarrow C_4 = -C_3 e^{2\lambda b}$

$$T(x,y) = C_1 \sin\frac{n\pi x}{a}\left[C_3 e^{\lambda y} - C_3 e^{2\lambda b} e^{-\lambda y}\right]$$

$$T(x,0) = f(x) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi x}{a}\underbrace{\left[1 - e^{\frac{2n\pi b}{a}}\right]}_{k}$$

for $f(x) = x$. $d_n = \frac{2}{k.a}\int_0^a x.\sin\frac{n\pi x}{a} = \frac{2}{ka}\left[\left(\frac{a}{n\pi}\right)^2 \sin\frac{n\pi x}{a} - \frac{x.a}{n\pi}\cos\frac{n\pi x}{a}\right]_0^a$

$$d_n = -\frac{2a}{\pi}\cdot\frac{\cos n\pi}{n\pi.\left(1 - e^{\frac{2n\pi b}{a}}\right)}$$

$$T(x,y) = -\frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{\left(1 - e^{\frac{2n\pi b}{a}}\right)}\frac{1}{n}\sin\frac{n\pi x}{a}\left[e^{\frac{n\pi y}{a}} - e^{\frac{n\pi}{a}(2b-y)}\right]$$

Ex 20: Vibrating String Problem.

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{a^2} \cdot \frac{\partial^2 U}{\partial t^2}$$

given $U(0,t) = U(L,t) = 0$ B.C

$U(x,0) = f(x)$, $\frac{\partial}{\partial t} U(x,0) = g(x)$ I.C

$U(x,t) = X(x) \cdot T(t)$

substitin eq. $\Rightarrow \frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{a^2 T} \frac{d^2 T}{dt^2} = -\lambda^2$ $\left.\begin{matrix} X(x) = C_1 \sin\lambda x + C_2 \cos\lambda x \\ T(t) = C_3 \sin\lambda at + C_4 \cos\lambda at. \end{matrix}\right\}$

① $U(0,t) = 0$; $X(0)\, T(t) = 0$; $X(0) = 0 \Rightarrow C_2 = 0$

② $U(L,t) = 0$; $X(L)\, T(t) = 0$; $X(L) = 0 \Rightarrow \lambda = \frac{n\pi}{L}$

$$U(x,t) = C_1 \sin\lambda_n x \left[C_3 \sin\lambda_n at + C_4 \cos\lambda_n at \right]$$

$$U(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \left[C_1 C_3 \sin\frac{n\pi at}{L} + C_1 C_4 \cos\frac{n\pi at}{L} \right]$$

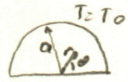
$$U(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \left[d_n \sin\frac{n\pi at}{L} + e_n \cos\frac{n\pi at}{L} \right]$$

③ $U(x,0) = f(x) = \sum_{n=1}^{\infty} e_n \sin\frac{n\pi x}{L} \Rightarrow e_n = \frac{2}{L} \int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$

④ $\frac{\partial}{\partial t} U(x,0) = g(x) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \left[d_n \cdot \left(\frac{n\pi a}{L}\right) \cancel{\cos\frac{n\pi a 0}{L}}^{1} + e_n \cdot 0 \right]$

$$d_n = \frac{\cancel{L}}{n\pi a} \cdot \frac{2}{\cancel{L}} \int_0^L g(x) \sin\frac{n\pi x}{L}\, dx$$

Ex 21: Temp. Dist In a Semi circular Slab. given $T(a,\theta) = T_0$, $T(r,0) = 0$, $T(r,\pi) = 0$



$$\nabla^2 T = \frac{1}{r^2}\left(r^2 \frac{\partial^2 T}{\partial r^2} + r\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \theta^2}\right) = 0$$

$$T(r,\theta) = R(r)\,\phi(\theta)$$

$$r^2 \frac{d^2R}{dr^2}\phi + r\frac{dR}{dr}\phi + \frac{d^2\phi}{d\theta}R = 0 \quad /* \frac{1}{R\phi} \Rightarrow \frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda^2$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{r}{R}\frac{dR}{dr} - \lambda^2 = 0 \Rightarrow R = r^m \Rightarrow R(r) = C_3 r^\lambda + C_4 r^{-\lambda}$$

$$\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} + \lambda^2 = 0 \Rightarrow \phi(\theta) = C_1 \sin\lambda\theta + C_2\cos\lambda\theta$$

①,② $T(r,0) = T(r,\pi) = 0 \Rightarrow \phi(0) = 0$ $\quad C_2 = 0$

$\phi(\pi) = 0$ $\quad C_1 \sin\lambda\pi = 0 = \sin n\pi \Rightarrow \lambda = n$

③ $T(a,\theta) = T_0$

$$T(r,\theta) = C_1 \sin(n\theta)\left[C_3 r^n + C_4 r^{-n}\right] = \sin(n\theta)\left[C_1C_3 r^n + C_1C_4 r^{-n}\right]$$

$n = \infty$

$\frac{1}{\infty} = \text{undef}$

$C_4 = 0$

$$T(r,\theta) = \sin(n\theta)\, C_1C_3 r^n$$

$$T(r,\theta) = \sum_{n=1}^{\infty} C_n \sin(n\theta) r^n$$

$$T(a,\theta) = T_0 = \sum_{n=1}^{\infty} C_n \sin n\theta\, a^n \Rightarrow C_n = \frac{1}{a^n}\frac{2}{\pi}\int_0^{\pi} T_0 \sin(n\theta)\, d\theta$$

$$= \frac{2T_0}{\pi . a^n}\left[-\frac{\cos(n\theta)}{n}\right]_0^{\pi} = \frac{2T_0}{n\pi . a^n}\cdot\left(1 - \cos n\pi\right)$$

$$T(r,\theta) = \frac{-2T_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\,a^n} r^n \left(\cos n\pi - 1\right)\sin(n\theta)$$

Ex 22. Wave Eq.

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 U}{\partial t^2}$$

$0<x<b \qquad 0<y<c \qquad 0<t<\infty$

given, $U(0,y,t) = U(b,y,t) = 0$ ① ②

$U(x,0,t) = U(x,c,t) = 0$ ③ ④

$U(x,y,0) = T.xy(x-b)(y-c)$ ⑤

$\frac{\partial}{\partial t}U(x,y,0) = 0$ ⑥

$U(x,y,t) = X.Y.T$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{a^2}\frac{1}{T}\frac{d^2T}{dt^2} \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{a^2}\frac{1}{T}\frac{d^2T}{dt^2} - \frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda^2$$

$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$

$$\frac{1}{a^2}\frac{1}{T}\frac{d^2T}{dt^2} - \frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda^2 \Rightarrow \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{a^2}\frac{1}{T}\frac{d^2T}{dt^2} + \lambda^2 = -\beta^2$$

$Y(y) = C_3 \cos\beta y + C_4 \sin\beta y.$

$$\left.\begin{aligned} &\frac{d^2T}{dt^2} + a^2\lambda^2 T + a^2\beta^2 T = 0 \\ &\frac{d^2T}{dt^2} + a^2(\lambda^2+\beta^2)T = 0 \end{aligned}\right\} T(t) = C_5 \cos(a\sqrt{\lambda^2+\beta^2}\,t) + C_6 \sin(a\sqrt{\lambda^2+\beta^2}\,t)$$

① $U(0,y,t) = 0 \Rightarrow X(0) = 0 \quad C_1 = 0$, ② $U(b,y,t) = 0 \Rightarrow X(b) = 0 \quad \lambda = \frac{n\pi}{b}$

③ $\rightarrow C_3 = 0$ ④ $\rightarrow \beta = \frac{m\pi}{c}$ $\Rightarrow$ $X(x) = C_2 \sin\frac{n\pi}{b}x$

$Y(y) = C_4 \sin\frac{m\pi}{c}y$

⑥ $\frac{\partial}{\partial t}U(x,y,0) = 0 \quad \frac{\partial T(0)}{\partial t} = 0 \Rightarrow C_6 = 0$

$$U = C_2 \sin\frac{n\pi x}{b} \cdot C_4 \sin\frac{m\pi y}{c} \cdot C_5 \cos\left(a\sqrt{\lambda^2+\beta^2}\,t\right)$$

$$U = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\left(\sin\frac{n\pi x}{b}\right)\left(\sin\frac{m\pi y}{c}\right)\cos\left(a\sqrt{\frac{n^2\pi^2}{b^2}+\frac{m^2\pi^2}{c^2}}\,t\right)$$

⑤ $U(x,y,0) = T x.y(x-b)(y-c) = \sum_n\sum_m C_{nm} \sin\frac{n\pi x}{b} \cdot \sin\frac{m\pi y}{c}$

$$C_{nm} = \frac{2}{b}\cdot\frac{2}{c}\int_{y=0}^{c}\int_{x=0}^{b} Txy(x-b)(y-c)\sin\frac{n\pi x}{b}\sin\frac{m\pi y}{c}\,dx\,dy$$

$$C_{nm} = \frac{16T\,b^2c^2}{\pi^6 n^3 m^3}\cdot(1-\cos m\pi)(1-\cos n\pi)$$

Ex 23 Heat Eqn.

$$\frac{\partial^2 U}{\partial x^2} - a^2 \frac{\partial U}{\partial t} = -Ae^{-\alpha x}$$

given $U(0,t) = 0$

$U(L,t) = 0$

$U(x,0) = f(x)$

$$U(x,t) = \Psi(x) + W(x,t)$$

$$\frac{d^2\Psi}{dx^2} + \frac{d^2 W}{dx^2} - a^2 \frac{\partial W}{\partial t} = -Ae^{-\alpha x}$$

① $U(0,t) = 0 \Rightarrow \Psi(0) + W(0,t) = 0$

$U(L,t) = 0 \Rightarrow \Psi(L) + W(L,t) = 0$

$U(x,0) = f(x) \Rightarrow \Psi(x) + W(x,0) = f(x)$

I

$$\frac{\partial^2 W}{\partial x^2} - a^2 \frac{\partial W}{\partial t} = 0$$

$W(0,t) = 0$

$W(L,t) = 0$

$W(x,0) = f(x) - \Psi(x)$

II

$$\frac{d^2\Psi}{dx^2} = -Ae^{-\alpha x}$$

$\Psi(0) = 0$

$\Psi(L) = 0$

Soln I $W(x,t) = X(x) \cdot T(t)$

$X(x) = D\sin\lambda + E\cos\lambda$

① $E = 0$

② $\lambda = \frac{n\pi}{L}$

$X(x) = D\sin\left(\frac{n\pi}{L}x\right) \Leftarrow$

$$T(t) = Fe^{-\frac{n^2\pi^2}{L^2 a^2}t}$$

$$W(x,t) = \sum_{n=1}^{\infty} G_n \sin\frac{n\pi x}{L} \cdot e^{-\frac{n^2\pi^2}{L^2 a^2}t}$$

Soln II $\Psi(x) = -\frac{A}{\alpha^2} e^{-\alpha x} + Bx + C$

$\Psi(0) = 0$; $\frac{A}{\alpha^2} + C = 0 \Rightarrow C = -\frac{A}{\alpha^2}$

$\Psi(L) = 0$; $\frac{A}{\alpha^2} e^{-\alpha L} + BL + \frac{A}{\alpha^2} = 0 \Rightarrow B = \frac{A}{\alpha^2 L}(e^{-\alpha L} - 1)$

$$\Psi(x) = -\frac{A}{\alpha^2} e^{-\alpha x} + \frac{A}{\alpha^2 L}(e^{-\alpha L} - 1)x + \frac{A}{\alpha^2}$$

$$W(x,0) = f(x) - \Psi(x) = \sum_{n=1}^{\infty} G_n \sin\frac{n\pi}{L}x$$

$$G_n = \frac{2}{L}\int_0^L [f(x) - \Psi(x)] \sin\frac{n\pi x}{L}\, dx$$

$$U(x,t) = \Psi(x) + W(x,t)$$

(2) A force function of two indep. variables. given $U(0,t) = U(L,t) = 0$

$$a^2 \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2} + h(x,t) \qquad U(x,0) = f(x) \qquad \frac{\partial}{\partial t} U(x,0) = g(x)$$

$U(x,t) = V(x,t) + W(x,t)$ $\qquad$ $h(x,t) = xt^2$

$$a^2 \frac{\partial^2 V}{\partial x^2} + a^2 \frac{\partial^2 W}{\partial x^2} = \frac{\partial^2 V}{\partial t^2} + \frac{\partial^2 W}{\partial t^2} + h(x,t)$$

		I	II
		$a^2 \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t^2} = 0$	$a^2 \frac{\partial^2 W}{\partial x^2} - \frac{\partial^2 W}{\partial t^2} = h(x,t)$
$U(0,t) = 0$	$V(0,t) + W(0,t) = 0 \Rightarrow$	$V(0,t) = 0$	$W(0,t) = 0$
$U(L,t) = 0$	$V(L,t) + W(L,t) = 0 \Rightarrow$	$V(L,t) = 0$	$W(L,t) = 0$
$U(x,0) = f(x)$	$V(x,0) + W(x,0) = f(x) \Rightarrow$	$V(x,0) = f(x)$	$W(x,0) = 0$
$\frac{\partial}{\partial t} U(x,0) = g(x)$	$\frac{\partial V}{\partial t}(x,0) + \frac{\partial}{\partial t} W(x,0) = g(x) \Rightarrow$	$\frac{\partial}{\partial t} V(x,0) = g(x)$	$\frac{\partial}{\partial t} W(x,0) = 0$

Soln to I

$V(x,t) = X(x)T(t)$ $\quad X(x) = C_1 \sin \frac{n\pi x}{L}$ $\qquad$ 1-1 $\Rightarrow C_2 = 0$ $\qquad$ 1-2 $\Rightarrow \lambda = \frac{n\pi}{L}$

$T(t) = C_3 \sin \frac{n\pi a t}{L} + C_4 \cos \frac{n\pi a t}{L}$

$$V(x,t) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \cdot \sin \frac{n\pi a t}{L} + D_n \sin \frac{n\pi x}{L} \cdot \cos \frac{n\pi a t}{L}$$

$$V(x,0) = f(x) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{L} \qquad D_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}$$

$$\frac{\partial}{\partial t} V(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} C_n \sin \frac{n\pi x}{L} \Rightarrow C_n = \frac{2}{L} \cdot \frac{L}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L}$$

Soln to II

Assume $\quad W(x,t) = \sum F(t) \sin \frac{n\pi x}{L} \xrightarrow{\text{Subst. in eq II}} -a^2 \sum_{n=1}^{\infty} F(t) \frac{n^2\pi^2}{L^2} \sin \frac{n\pi x}{L} - \sum_{n=1}^{\infty} \frac{d^2F}{dt^2} \sin \frac{n\pi x}{L} = xt^2$

$$\sum_{n=1}^{\infty} \underbrace{\left[\frac{-a^2 n^2 \pi^2}{L^2} F(t) - \frac{d^2 F(t)}{dt^2} \right]}_{a_n(t)} \sin \frac{n\pi x}{L} = xt^2$$

$$a_n(t) = \frac{2}{L} \int_0^L xt^2 \sin \frac{n\pi x}{L} dx \Rightarrow a_n(t) = \frac{-2(-1)^n}{n\pi} L t^2$$

$$\frac{d^2F}{dt^2} + \frac{a^2 n^2 \pi^2}{L^2} F(t) = \frac{2(-1)^n L}{n\pi} t^2$$

$F_p(t) = C_1 t^2 + C_2 t + C_3 \Rightarrow C_1 = \frac{2(-1)^n}{\pi n} L \left(\frac{L}{an\pi}\right)^2 \quad C_2 = 0$

$C_3 = \frac{-4(-1)^n}{\pi n} L \left(\frac{L}{an\pi}\right)^4$

$$F_H(t) = D \sin \frac{n\pi a}{L} t + E \cos \frac{n\pi a t}{L}$$

$$F(t) = F_h + F_p = D \sin \frac{n\pi a t}{L} + E \cos \frac{n\pi a t}{L} + \frac{2(-1)^n}{\pi n} L \left(\frac{L}{an\pi}\right)^2 t^2 - \frac{4(-1)^n}{\pi n} L \left(\frac{L}{an\pi}\right)^4$$

$W(x,0) = 0 \Rightarrow \sum F(0) \sin \frac{n\pi x}{L} = 0 \Rightarrow F(0) = 0 \Rightarrow E = \frac{4(-1)^n}{n\pi} L \left(\frac{L}{n\pi a}\right)^4$, $\quad \frac{dF(0)}{dt} = 0 \Rightarrow D = 0$

$$F(t) = \frac{4(-1)^n}{n\pi} L \left(\frac{L}{n\pi a}\right)^4 \cos \frac{n\pi a t}{L} + \frac{2(-1)^n}{n\pi} L \left(\frac{L}{an\pi}\right)^2 - \frac{4(-1)^n}{} L \left(\frac{L}{}\right)^4$$

Ex 26.

$$a^2 \frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2} + h(x,t)$$

$U(0,t) = \mu_1(t)$ $\quad U(x,0) = f(x)$

$U(L,t) = \mu_2(t)$ $\quad \frac{\partial}{\partial t} U(x,0) = g(x)$

$U(x,t) = V(x,t) + W(x,t)$

$$a^2 \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t^2} = \underbrace{-a^2 \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial t^2} + h(x,t)}_{H(x,t)}$$

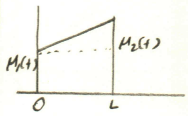
$$a^2 \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial t^2} = H(x,t)$$

$U(0,t) = \mu_1(t)$; $V(0,t) = 0$, $\quad W(0,t) = \mu_1(t)$

$U(L,t) = \mu_2(t)$, $V(L,t) = 0$, $\quad W(L,t) = \mu_2(t)$

$U(x,0) = f(x)$ $\quad V(x,0) = f(x) - W(x,0) = F(x)$

$\frac{\partial}{\partial t} U(x,0) = g(x)$ $\quad \frac{\partial}{\partial t} V(x,0) = g(x) - \frac{\partial}{\partial t} W(x,0) = G(x)$



$$W(x,t) = \mu_1(t) + [\mu_2(t) - \mu_1(t)]\frac{x}{L}$$

$V(x,t) = \psi(x,t) + \phi(x,t)$

① $a^2 \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial t^2} = 0 \Rightarrow \psi(x,t) = X(x)\,T(t)$

② $a^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = H(x,t) \Rightarrow \phi(x,t) = \sum F(t) \sin\frac{n\pi x}{L}$

$$\psi(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} \cdot \sin\frac{n\pi a t}{L} + D_n \sin\frac{n\pi x}{L} \cdot \cos\frac{n\pi a t}{L}$$

$$D_n = \frac{2}{L}\int_0^L F(x) \sin\frac{n\pi x}{L}\,dx \,, \quad C_n = \frac{2}{n\pi a}\int_0^L G(x) \sin\frac{n\pi x}{L}\,dx$$

$$\sum_{n=1}^{\infty} \Big[\underbrace{-\frac{a^2 n^2 \pi^2}{L^2} F(t) - \frac{d^2 F(t)}{dt^2}}_{d_n(t)}\Big] \sin\frac{n\pi x}{L} = H(x,t) \qquad d_n(t) = \frac{2}{L}\int_0^L H(x,t) \sin\frac{n\pi x}{L}\,dx.$$

$$\frac{d^2 F}{dt^2} + \frac{a^2 n^2 \pi^2}{L^2} F(t) = -d_n(t) \Rightarrow F(t) = F_H + F_P$$

$U = V + W$

$U = \psi + \phi + W$

$U = X \cdot T + \sum F_t \sin\frac{n\pi x}{L} + W$

Ex 27. $\dfrac{\partial^2 U}{\partial x^2} = \dfrac{1}{k}\dfrac{\partial U}{\partial t}$ given $U(0,t) = 0$, $U(L,t) = \frac{1}{2} k L t^2$, $U(x,0) = 0$

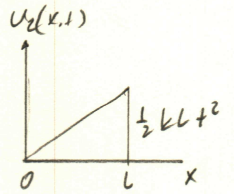
$U(x,t) = U_1(x,t) + U_2(x,t)$

subst in Eq.

$$\frac{\partial^2 U_1}{\partial x^2} - \frac{1}{k}\frac{\partial U_1}{\partial t} = \underbrace{-\frac{\partial^2 U_2}{\partial x^2} + \frac{1}{k}\frac{\partial U_2}{\partial t}}_{H(x,t)}$$

$U_1(0,t) = 0$

$U_1(L,t) = 0$



$U_2(0,t) = 0$

$U(x,0) = 0 \Rightarrow U_1(x,0) = -U_2(x,0) = 0$ $U_2(L,t) = \frac{1}{2} k L t^2$

$U_2(x,t) = \frac{1}{2} k x t^2$

$U_1(x,t) = \sum T(t) \sin \frac{n\pi x}{L}$

$$\Rightarrow -\sum_{n=1}^{\infty} T(t)\left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L} - \frac{1}{k}\sum_{n=1}^{\infty}\frac{dT(t)}{dt}\sin\frac{n\pi x}{L} = xt$$

$$\sum_{n=1}^{\infty}\left[\frac{1}{k}\frac{dT}{dt} + \left(\frac{n\pi}{L}\right)^2 T\right]\sin\frac{n\pi x}{L} = -xt$$

$$A_n(t) = -\frac{2}{L}\int_0^L xt \sin\frac{n\pi x}{L}\,dx = -\frac{2t}{L}\int_0^L x \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2L}{n\pi}\cos n\pi \cdot t = \frac{1}{k}\frac{dT}{dt} + \left(\frac{n\pi}{L}\right)^2 T$$

$T_H(t) = C e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ $T_p = D_1 + D_2$. $D_1 = \dfrac{2L^3}{n^3\pi^3}\cos n\pi$

$D_2 = -\dfrac{2}{k}\dfrac{L^5}{n^5\pi^5}\cos n\pi$

$$T(t) = C e^{-k\left(\frac{n\pi}{L}\right)^2 t} + 2\left(\frac{L}{n\pi}\right)^3(\cos n\pi)\,t - \frac{2}{k}\left(\frac{L}{n\pi}\right)^5 \cos n\pi$$

$U(x,0) = 0 \Rightarrow C - \dfrac{2}{k}\left(\dfrac{L}{n\pi}\right)^5 \cos n\pi = 0 \Rightarrow C = \dfrac{2}{k}\left(\dfrac{L}{n\pi}\right)^5 \cos n\pi$

$U(x,t) = U_1(x,t) + U_2(x,t) =$

$$\sum\left[\frac{2}{k}\left(\frac{L}{n\pi}\right)^5 \cos n\pi\, e^{-k\left(\frac{n\pi}{L}\right)^2 t} + 2\left(\frac{L}{n\pi}\right)^3 \cos n\pi\,(t) - \frac{2}{k}\left(\frac{L}{n\pi}\right)^5 \cos n\pi\right]\sin\frac{n\pi x}{L} + \frac{1}{2} k t^2 x$$

Ex 28 : 2D Heat Problem.

$$c^2\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right)=\frac{\partial U}{\partial t} \qquad 0\le x\le a \quad 0\le y\le b$$

$U(0,y,t) = U(a,y,t) = 0$
$U(x,0,t) = U(x,b,t) = 0$
$U(x,y,0) = T_0$

$U(x,y,t) = X(x)\cdot Y(y)\cdot T(t)$

$$\frac{\partial^2 X}{\partial x^2}YT+\frac{\partial^2 Y}{\partial y^2}XT=\frac{1}{c^2}\frac{\partial U}{\partial t}XY$$

$\frac{\partial T}{\partial t} = -\lambda^2 c^2 T$

$C e^{-\lambda c}$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=\frac{1}{c^2 T}\frac{\partial T}{\partial t}=-\lambda^2$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=-\lambda^2$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=\alpha^2-\lambda^2$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=\underbrace{-\lambda^2-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-\alpha^2}$$

$X(x) = C_1 \cos\alpha x + C_2 \sin\alpha x$ 1, 2. $C_1 = 0$, $\alpha = \frac{n\pi}{a}$ $X(x) = C_2 \sin\frac{n\pi x}{a}$

$Y(y) = C_3\cos\sqrt{\alpha^2-\lambda^2}\,y + C_4 \sin\sqrt{\alpha^2-\lambda^2}\,y$ $C_3 = 0$ $C_4 \sin\sqrt{\alpha^2-\lambda^2}\,b = 0 = \sin(n\pi)$

$\sqrt{\alpha^2-\lambda^2} = \frac{n\pi}{b}$

$Y(y) = C_4 \sin\frac{n\pi y}{b}$

$$U(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} d_{nm}(t)\,\sin\frac{n\pi x}{a}\,\sin\frac{m\pi y}{b}\; e^{-c^2\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)t}$$

$$d_{mn}=\frac{4}{ab}\int_0^b\int_0^a T_0 \sin\frac{m\pi}{a}x\,\sin\frac{n\pi}{b}y\,dx\,dy.$$